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Abstract - In this paper the asymptotic behavior of the discrete minimum variance output feedback control law is investigated. It is shown that this control law results in an asymptotically stable closed-loop system. For time invariant systems, the output feedback gain tends to a periodic function of time which is easier to implement in the feedback loop.

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I. INTRODUCTION

Consider the stochastic linear time-varying system S_c^t whose output can only be measured at discrete instants of time and is given by

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) + D(t)w(t) \tag{1}$$

$$y(t_i) = H(t_i)x(t_i) + v(t_i)$$

where F(t), G(t), and D(t) are nxn, nxr, and nxp continuous matrices; w(t), the disturbance in the dynamics is a stochastic process with a gaussian distribution whose first- and second-order statistics are given by

$$E[w(t)] = 0; \quad E[w(t)w'(\tau)] = \bar{Q}(t)\,\delta\,(t - \tau) \tag{2}$$

where $\bar{Q}(t)$ is a continuous positive semidefinite pxp matrix and $\delta(.)$ is the Dirac delta function. Similarly $[v(t_i), \ i = 0, 1, \ldots]$ is an m-dimensional gaussian white sequence for which

$$E\,[v(t_i)] = 0; \; E\,[v(t_i)v(t_j)] = R(t_i)\,\delta_{ij} \tag{3}$$

where $R(t_i)$ is a positive definite mxm matrix. The initial state $x(t_o)$ of system S_c^t is assumed to be a random variable for which

$$E\,[x(t_o)] = 0; \; E\,[x(t_o)x'(t_o)] = \Sigma_o \tag{4}$$

Furthermore assume that for all $t \geq t_o$ and t_i

$$E\left[w(t)v'(t_i)\right] = E\left[w(t)x'(t_o)\right] = E\left[v(t_i)\ x'(t_o)\right] = 0 \qquad (5)$$

Let the class of admissible controls U_c, be the set of linear functions of the current output measurements, which is given by

$$U_c = \left[u \in R^r = u[t, y(t_i)]\right] = C_i(t)y(t_i) \ ; \ t_i \leq t < t_{i+1}, i=0,1,\ldots N-1 \qquad (6)$$

Then it was shown in reference [1] that the control law

$$u^*(t,y(t_i)) = -G'(t)\,\Phi'(t_i,t)\ W^{-1}(t_i,t_{i+1})\ \Sigma^*(t_i)H'(t_i)[H(t_i)$$
$$\Sigma^*(t_i)H'(t_i) + R(t_i)]^{-1}\ y(t_i) \qquad (7)$$

where

$$\Sigma^*(t_{i+1}) = \Phi(t_{i+1},t_i)\left[\Sigma^*(t_i) - \Sigma^*(t_i)H'(t_i)\left[H(t_i)\,\Sigma^*(t_i)H'(t_i)\right.\right.$$
$$\left.\left. + R(t_i)\right]^{-1} H(t_i)\,\Sigma^*(t_i)\right]\Phi'(t_{i+1},t_i) + Q(t_{i+1},t_i) \qquad (8)$$

with

$$\Sigma^*(t_o) = \Sigma_o \ ; \quad Q(t_{i+1},t_i) = \int_{t_i}^{t_{i+1}} \Phi(t_{i+1},t)D(t)\,\bar{Q}(t)\,D'(t)\Phi'(t_{i+1},t)dt$$

and

$$W(t_i,t_{i+1}) = \int_{t_i}^{t_{i+1}} \Phi(t_i,t)\ G(t)\ G'(t)\ \Phi'(t_i,t)\ dt \qquad (9)$$

minimizes the cost function

$$J(u) = 1/2\ E\left[x'(t_{i+1})\ S'_{i+1}\ S_i\ x(t_{i+1})\right] \qquad i = 0,\ 1,\ \ldots,\ N-1 \qquad (10)$$

subject to the constraints and assumptions (1) - (6), provided that $W(t_i,t_{i+1})$ is invertible.

The objective of this paper is to present some of the asymptotic properties of the discrete minimum variance output feedback control law (7) and to show that the resultant closed-loop system is asymptotically stable in the sense of Lyapunov. It should be noted however, that the results developed in this paper, make use of the properties of the covariance equation (8) of the closed-loop system, which have been widely investigated in the literature [2], [3], [4] , and [5], in connection with the Kalman filter. Before proceeding, it will be assumed throughout the following that the assumptions

(A.1) $$\|\Phi(t,s)\| \leq \alpha_3(|t - s|) \qquad \forall t \text{ and } s \tag{11}$$

(A.2) $$0 < \alpha_1 I \leq W(t_i,t_{i+1}) \leq \alpha_2 I < \infty \qquad \forall i \tag{12}$$

(A.3) $$0 < \beta_1 I \leq \sum_{i=k-N}^{k-1} \Phi(t_k,t_{i+1}) Q(t_i,t_{i+1}) \Phi'(t_k,t_{i+1}) \leq \beta_2 I < \infty \qquad \forall k \geq N \tag{13}$$

(A.4) $$0 < \gamma_1 I \leq \sum_{i=k-N}^{k} \Phi'(t_i,t_k) H'(t_i) R^{-1}(t_i) H(t_i) \Phi(t_i,t_k) \leq \gamma_2 I < \infty \qquad \forall k \geq N \tag{14}$$

are satisfied for some real α_1, α_2, α_3, β_1, β_2, γ_1, and γ_2.

II. STABILITY OF THE DISCRETE MINIMUM-VARIANCE OUTPUT FEEDBACK CONTROL SYSTEM

In this section the asymptotic stability of the discrete time minimum variance closed-loop system will be investigated in the case when no noise or disturbances are present in the measurements or the dynamics. To be specific, let the system under consideration be given by

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) \tag{15}$$

$$y(t_i) = H(t_i)x(t_i)$$

and let $u(t)$ be given by

$$u(t) = u^*(t,y(t_i)) = -G'(t)\,\Phi'(t_i,t)W^{-1}(t_i,t_{i+1})\,\Sigma^*(t_i)H'(t_i)$$

$$\left[H(t_i)\,\Sigma^*(t_i)H'(t_i) + R(t_i)\right]^{-1} y(t_i) \tag{16}$$

For this control law the state at the sampling times is given by

$$x(t_{i+1}) = \Phi(t_{i+1},t_i)\,\{I - \Sigma^*(t_i)H'(t_i)\left[H(t_i)\,\Sigma^*(t_i)H'(t_i)\right.$$

$$\left. + R(t_i)\right]^{-1}H(t_i)\,\}\,x(t_i) \tag{17}$$

The objective of this section is to show that starting from any initial condition $x(t_o)$, the state of the closed-loop system (17), $x(t_i)$, tends to zero as t_i approaches infinity.

Let $P^*(t_{i+1})$ be given by

$$P^*(t_{i+1}) = \{\Sigma^*(t_i) - \Sigma^*(t_i)H'(t_i)[H(t_i)\Sigma^*(t_i)H'(t_i)$$

$$+ R(t_i)]^{-1}H(t_i)\Sigma^*(t_i)\} \qquad (18)$$

Then it was shown in references [2] and [3] that $P^*(t_{i+1})$ is bounded from above and below for every i, provided assumptions (A.3) and (A.4) are satisfied. These results are summarized in the following lemma.

Lemma 1: (Deyst and Price) If assumptions (A.3) and (A.4) are satisfied, then $P^*(t_{i+1})$ is bounded from above and below for every i by

$$\left\{\frac{\beta_1}{1+\beta_1\gamma_1}\right\}I \le P^*(t_{i+1}) \le \left\{\frac{1}{\gamma_2} + \frac{N\gamma_1^2\beta_2}{\gamma_2^2}\right\}I \qquad (19)$$

Theorem 1: Assume that (A.1) through (A.4) are satisfied, then the discrete time closed-loop system (17) is uniformly asymptotically stable in the large and

$$V[x(t_{i+1}),t_{i+1}] = x'(t_{i+1})\,\Phi'(t_i,t_{i+1})P^{*^{-1}}(t_{i+1})\Phi(t_i,t_{i+1})x(t_{i+1}) \qquad (20)$$

is a Lyapunov function, where $P^*(t_{i+1})$ is given (18)

Proof: Using assumption (A.1), it can be easily deduced that

$$\alpha_3^{-2}(|t_{i+1} - t_i|) \le \Phi'(t_i,t_{i+1})\Phi(t_i,t_{i+1}) \le \alpha_3^2(|t_{i+1} - t_i|) \qquad (21)$$

Furthermore, from (21) and (19) it can be deduced that for all t_i

$$\left\{ \frac{\gamma_2^2 \alpha_3^{-2}}{\gamma_2 + N\gamma_1^2\beta_2} \right\} I \leq \Phi'(t_i, t_{i+1}) P^{-1}(t_{i+1}) \Phi(t_i, t_{i+1}) \leq$$

$$\left\{ \frac{(1 + \beta_1 \gamma_1)}{\beta_1} \alpha_3^2 \right\} I \tag{22}$$

Hence $V[x(t_{i+1}), t_{i+1}]$ is uniformly bounded from above and below for every t_i, i.e., there exist continuous nondecreasing scalar functions $\delta_1[\|x(t_{i+1})\|]$ and $\delta_2[\|x(t_{i+1})\|]$ such that $\delta_1(0) = \delta_2(0) = 0$ and

$$0 < \delta_1[\|x(t_{i+1})\|] \leq V[x(t_{i+1}), t_{i+1}] \leq \delta_2[\|x(t_{i+1})\|]$$

$$< \infty \ \forall t_i \tag{23}$$

Hence (20) is a Lyapunov function. To show that (17) is uniformly asymptotically stable in the large, we need to show [6], that for some finite N, there exists a negative definite function $\delta_3[\|x(t_{i+1})\|]$ such that $\forall t_i > t_N$

$$V[x(t_{i+1}), t_{i+1}] - V[x(t_{i-N}), t_{i-N}] \leq \delta_3[\|x(t_{i+1})\|] < 0 \tag{24}$$

Consider the function

$$V[x(t_{i+1}), t_{i+1}] = x(t_{i+1}) \Phi'(t_i, t_{i+1}) P^{*^{-1}}(t_{i+1}) \Phi(t_i, t_{i+1}) x(t_{i+1})$$

Substituting for $x(t_{i+1})$ from (17) we get

$$V[x(t_{i+1}),t_{i+1}] = x'(t_i)\Sigma^{*^{-1}}(t_i)P^*(t_{i+1})\Sigma^{*^{-1}}(t_i)x(t_i)$$

$$= x'(t_i)\Sigma^{*^{-1}}(t_i)x(t_i) - x'(t_i)H'(t_i)\left[H(t_i)\Sigma^*(t_i)\right.$$

$$\left.H'(t_i) + R(t_i)\right]^{-1}H(t_i)x(t_i) \tag{25}$$

Since

$$\Sigma^*(t_i) = \Phi(t_i,t_{i-1})P^*(t_i)\Phi'(t_i,t_{i-1}) + Q(t_i,t_{i-1}) \tag{26}$$

it follows that

$$\Sigma^*(t_i) \geq \Phi(t_i,t_{i-1})P^*(t_i)\Phi'(t_i,t_{i-1}) \tag{27}$$

from which it can be deduced that

$$\Phi'(t_{i-1},t_i)P^{*^{-1}}(t_i)\Phi(t_{i-1},t_i) \geq \Sigma^{*^{-1}}(t_i) \tag{28}$$

From (28) and (25) we have

$$V[x(t_{i+1}),t_{i+1}] \leq V[x(t_i),t_i] - x'(t_i)H'(t_i)\left[H(t_i)\Sigma^*(t_i)H'(t_i)\right.$$

$$\left.+ R(t_i)\right]^{-1}H(t_i)x(t_i) \tag{29}$$

Repeating the above for k = i-1, ..., i-N we get

$$V[x(t_{i+1}),t_{i+1}] - V[x(t_{i-N}),t_{i-N}] \le - \sum_{k=i}^{i-N} x'(t_k)H'(t_k)\,[H(t_k)\,\Sigma^*(t_k)$$

$$H'(t_k) + R(t_k)\,]^{-1}\,H(t_k)x(t_k) \tag{30}$$

From (26) we have

$$[R(t_k) + H(t_k)\,\Sigma^*(t_k)H'(t_k)\,] = \{\,R(t_k) + H(t_k)\,[\,\Phi\,(t_k,t_{k-1})P^*(t_k)$$

$$\Phi'(t_k,t_{k-1}) + Q\,]H'(t_k)\} \tag{31}$$

and since $\Phi(t_k,t_{k-1})P^*(t_k)\Phi'(t_k,t_{k-1})$ is bounded from above for all k, it can be deduced from (31) that there exists an $\eta > 0$ such that

$$[R(t_k) + H(t_k)\,\Sigma^*(t_k)H'(t_k)] \le [\,R(t_k) + \eta H(t_k)H'(t_k)\,] < \infty \tag{32}$$

Using (32) in inequality (30) we get

$$V[x(t_{i+1}),t_{i+1}] - V[x(t_{i-N}),t_{i-N}] \le -\sum_{k=i}^{i-N} x'(t_k)H'(t_k)\,[R(t_k)$$

$$+ \eta H(t_k)H'(t_k)]^{-1}\,H(t_k)x(t_k) \tag{33}$$

Since (A.4) implies that (15) is discrete time observable and since observability is invariant under output feedback control laws, it follows that the right hand side of (33) is strictly less than zero. Hence there exists a $\delta_3[\,\|x(t_{i+1})\|\,]$ such that (24) is satisfied, from which it can be concluded that (17) is asymptotically stable in the large.

QED

Remark 1: Assumptions (A.1) and (A.2) are equivalent to the assumption that system (17) is uniformly completely controllable and is needed to ensure the uniform boundedness of the control law (16) for all t, and that of $W(t_i, t_{i+1})$ for all i.

III. ASYMPTOTIC BEHAVIOR OF THE DISCRETE MINIMUM - VARIANCE OUTPUT FEEDBACK CONTROL LAW

In this section the steady state behavior of the discrete minimum - variance output feedback control law (16) is considered. The results developed make use of the properties of the discrete matrix Riccati equation, which has been widely investigated in the literature [2] - [5].

Let $\Sigma(t_i)$ be the resulting state covariance of the closed-loop system for any admissible control

$$u(t,y(t_i)) = \bar{K}(t)\, y(t_i) \tag{34}$$

Then the recursive relation that determines $\Sigma(t_i)$, is given by

$$\Sigma(t_{i+1}) = \Phi(t_{i+1},t_i)\,[I + K(t_i)H(t_i)]\,\Sigma(t_i)\,[I + K(t_i)H(t_i)]'$$

$$\Phi'(t_{i+1},t_i) + \Phi(t_{i+1},t_i)K(t_i)R(t_i)K'(t_i)\,\Phi'(t_{i+1},t_i) + Q(t_i,t_{i+1}) \tag{35}$$

where

$$\Sigma(t_o) = \Sigma_o \tag{36}$$

and

$$K(t_i) = \int_{t_i}^{ti+1} \Phi(t_i,t)\ G(t)\bar{K}(t)\ dt \tag{37}$$

Lemma 2: Assume that

$$W(t_i,t_{i+1}) > 0 \tag{38}$$

Then the control law $u^*(t, y(t_i))$as given by (16) minimizes the state covariance of the closed-loop system at the sampling time.

Proof: We would like to show that for any $u(t, y(t_i)) \epsilon U_c$

$$\Sigma(t_{i+1}) \geq \Sigma^*(t_{i+1}) \qquad i = 0,1,\ldots,N-1 \tag{39}$$

where $\Sigma(t_i)$ and $\Sigma^*(t_i)$ are given by (35) and (8) respectively. Assume for now that for some i

$$\Sigma(t_i) \geq \Sigma^*(t_i) \tag{40}$$

and let $K^*(t_i)$ be given by

$$K^*(t_i) = \Sigma^*(t_i)H'(t_i)[H'(t_i)\Sigma^*(t_i)H(t_i) + R(t_i)]^{-1} \tag{41}$$

then it follows from assumption (40) that

$$\Sigma(t_{i+1}) \geq \Phi(t_{i+1},t_i) [I + K(t_i)H(t_i)] \Sigma^*(t_i) [I + K(t_i)H(t_i)]'$$

$$\Phi'(t_{i+1},t_i) + \Phi(t_{i+1},t_i)K(t_i)R(t_i)K'(t_i)\Phi'(t_{i+1},t_i) + Q(t_i,t_{i+1})$$

$$\geq \Sigma^*(t_{i+1}) + \Phi(t_{i+1},t_i) \Sigma^*(t_i)H'(t_i) [H(t_i) \Sigma^*(t_i)H'(t_i)$$

$$+ R(t_i)]^{-1}H(t_i) \Sigma^*(t_i) \Phi'(t_{i+1},t_i) + \Phi(t_{i+1},t_i)K(t_i)[H(t_i)$$

$$\Sigma^*(t_i)H'(t_i) + R(t_i)]K'(t_i)\Phi'(t_{i+1},t_i)$$

$$+ \Phi(t_{i+1},t_i)K(t_i)H(t_i) \Sigma^*(t_i) \Phi'(t_{i+1},t_i)$$

$$+ \Phi(t_{i+1},t_i) \Sigma^*(t_i)H'(t_i)K'(t_i) \Phi'(t_{i+1},t_i)$$

$$\geq \Sigma^*(t_{i+1}) + \Phi(t_{i+1},t_i) [K(t_i) + K^*(t_i)] [H(t_i) \Sigma^*(t_i)H'(t_i)$$

$$+ R(t_i)] [K(t_i) + K^*(t_i)]' \Phi'(t_{i+1},t_i)$$

$$\geq \Sigma^*(t_{i+1}) \tag{42}$$

Since $\Sigma(t_o) = \Sigma^*(t_o) = \Sigma_o$, assumption (40) is satisfied for t_o, and it follows from (42) that for any $u[t, y(t_i)] \in U_c$

$$\Sigma(t_1) \geq \Sigma^*(t_1)$$

and (39) follows by induction on i.

QED

It should be noted that lemma 1 is a generalization of the results presented in reference [1] , where it was shown that $u^*[t, y(t_i)]$ minimizes the trace of the state covariance at the sampling times.

A basic assumption that was used in showing the optimality of $u^*[t,y(t_i)]$ is the a priori knowledge of the initial covariance of the state $\Sigma(t_o)$. In the sequel it will be shown that if assumptions (A.1) through (A.4) are satisfied, then the steady state behavior of $\Sigma^*(t_i)$ is independent of $\Sigma(t_o)$.

Let $\Sigma^*(t_i, t_o, \Sigma_o)$ denote the optimal value of the state covariance matrix, starting from the initial value $\Sigma^*(t_o) = \Sigma_o$, and let $\Phi_c^*(t_{i+1}, t_i, \Sigma_o)$ given by

$$\Phi_c^*(t_{i+1}, t_i, \Sigma_o) = \Phi(t_{i+1}, t_i)\, [I - K^*(t_i, \Sigma_o)H(t_i)] \tag{43}$$

Where $K^*(t_i, \Sigma_o)$ is given by (41).

Then by simple matrix manipulations it can be shown that

$$\Phi_c^*(t_{i+1}, t_i, 0)\, [\Sigma^*(t_i, t_o, \Sigma_o) - \Sigma^*(t_i, t_o, 0)]\, \Phi_c^{*\prime}(t_{i+1}, t_i, 0)$$

$$= \Sigma^*(t_{i+1}, t_o, \Sigma_o) - \Sigma^*(t_{i+1}, t_o, 0) + [K^*(t_i, \Sigma_o) - K^*(t_i, 0)]$$

$$[H(t_i)\, \Sigma^*(t_i, t_o, 0)H'(t_i) + R(t_i)]\, [K^*(t_i, \Sigma_o) - K^*(t_i, 0)]' \tag{44}$$

from which it can be deduced that

$$\Phi_c^*(t_{i+1},t_i,0)\ \{\Sigma^*(t_i,t_o,\Sigma_o) - \Sigma^*(t_i,t_o,0\}\ \Phi_c^{*\prime}(t_{i+1},t_i,0)$$

$$\geq \Sigma^*(t_{i+1},t_o,\Sigma_o) - \Sigma^*(t_{i+1},t_o,0) \tag{45}$$

Applying the above inequality repeatedly we get

$$\Phi_c^*(t_{i+1},t_o,0)\ \Sigma_o\,\Phi_c^{*\prime}(t_{i+1},t_o,0)\ \geq \Sigma^*(t_{i+1},t_o,\Sigma_o) - \Sigma^*(t_{i+1},t_o,0)$$

Since assumptions (A.1) - (A.4) are satisfied it follows from theorem 1, that there exist positive real numbers α_1 and α_2 such that

$$\|\Phi_c^*(t_{i+1},t_o,0)\| \leq \alpha_1\, e^{-\alpha_2 |t_{i+1} - t_o|} \tag{46}$$

Hence

$$\lim_{t_o \to -\infty} \{\Sigma^*(t_{i+1},t_o,\Sigma_o) - \Sigma^*(t_{i+1},t_o,0\} \leq 0 \tag{47}$$

But since $\Sigma_o \geq 0$, it follows from lemma 2 that

$$\Sigma^*(t_{i+1},t_o,\Sigma_o) - \Sigma^*(t_{i+1},t_o,0) \geq 0 \quad \forall\, t_i \text{ and } t_o \tag{48}$$

From (48) and (47) it can be deduced that

$$\lim_{t_o \to -\infty} \{\Sigma^*(t_{i+1}, t_o, \Sigma_o) - \Sigma^*(t_{i+1}, t_o, 0)\} = 0 \tag{49}$$

Since $\Sigma^*(t_{i+1}, t_o, \Sigma_o)$ and $\Sigma^*(t_{i+1}, t_o, 0)$ are bounded from above and below it follows that

$$\lim_{t_o \to -\infty} \Sigma^*(t_{i+1}, t_o, \Sigma_o) = \lim_{t_o \to -\infty} \Sigma^*(t_{i+1}, t_o, 0) \triangleq \tilde{\Sigma}^*(t_{i+1}) \tag{50}$$

Applying the above results to the problem under consideration we get:

<u>Theorem 2</u>: If assumptions (A.1)-(A.4) are satisfied then the output feedback control law

$$u^*(t, y(t_i)) = - G'(t)\, \Phi'(t_i, t)\, W^{-1}(t_i, t_{i+1})\, \tilde{\Sigma}^*(t_i) H'(t_i) \{H(t_i) \tilde{\Sigma}^*(t_i) H'(t_i) + R(t_i)\}^{-1} y(t_i) \tag{51}$$

minimizes

$$\lim_{t_o \to -\infty} E\{x(t_{i+1}) x'(t_{i+1})\} \triangleq \lim_{t_o \to -\infty} \Sigma(t_{i+1}, t_o, \Sigma_o)$$

subject to the constraints (1) - (6).

Where

$$\tilde{\Sigma}^*(t_{i+1}) = \Phi(t_{i+1},t_i)\ \{\tilde{\Sigma}^*(t_i) - \tilde{\Sigma}^*(t_i)H(t_i)\ \{H(t_i)\tilde{\Sigma}^*(t_i)H'(t_i)$$

$$+ R(t_i)\}^{-1}H'(t_i)\ \tilde{\Sigma}^*(t_i)\}\,\Phi'(t_{i+1},t_i) + Q(t_{i+1},t_i) \qquad (52)$$

$$\tilde{\Sigma}^*(t_o) = 0$$

In the time invariant case where the output measurements are sampled at a constant interval T, the matrices F, G, H, Q and R are independent of time, it can be shown [4], that

$$\lim_{t_o \to -\infty} \Sigma^*(t_{i+1},t_o\,\Sigma_o) = \lim_{t_o \to -\infty} \Sigma^*(t_{i+1},t_o,0) = \tilde{\Sigma}^* \qquad (53)$$

where $\tilde{\Sigma}^*$ is the positive definite solution to the equation

$$\tilde{\Sigma}^* = \Phi(T)\ \{\tilde{\Sigma}^* - \tilde{\Sigma}^*H\ \{H\ \tilde{\Sigma}^*H' + R\}^{-1}H'\ \tilde{\Sigma}^*\}\,\Phi'(T) + Q \qquad (54)$$

In this special case assumptions (A.1) - (A.4) reduce to

(A.1)' $\{F, G\}$ is a controllable pair

(A.2)' $\{\Phi(T),H\}$ is an observable pair

(A.3)' $\{\Phi(T),C\}$ is controllable, where $Q = CC'$

and we have the following theorem:

<u>Theorem 3</u>: Assume that (A.1)' - (A.3)' are satisfied, then the periodic output feedback control law

$$u^*(t, y(kT)) = - G' \Phi '(t, kT) W^{-1}(T) \tilde{\Sigma}^* H' \{ H \tilde{\Sigma}^* H' + R \}^{-1} y(kT)$$

$$kT \le t < (k+1) T, \; k = k_o, k_o + 1, \ldots \tag{55}$$

minimizes

$$\lim_{k_o \to -\infty} E \{ x(kT) x'(kT) \} \triangleq \lim_{k_o \to -\infty} \Sigma (kT, k_o, \Sigma_o) \; ; \tag{56}$$

where

$$W(T) = \int_o^T e^{-Ft} \, G \, G' \, e^{-F't} \, dt \tag{57}$$

and $\tilde{\Sigma}^*$ is the positive definite solution of the matrix equation (54).

Furthermore the resultant closed-loop system is asymptotically stable in the sense of Lyapunov.

The proof of the above theorem is similar to that of theorem 2 and hence will be omitted.

IV. CONCLUSION

The discrete minimum variance output feedback control law considered in this paper results in an asymptotically stable closed-loop system. Hence besides its optimality, this control law can be used in stabilizing linear systems by output feedback. The assumptions that were made in establishing the asymptotic properties of the control law are standard ones. Under these assumptions, the control law minimizes the steady state value of the state covariance at the sampling times, regardless of the assumed value of the system initial state covariance. In the time invariant case, the output feedback gain approaches a periodic function of time, which is easier to compute and implement.

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